



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

(1) The Parallel-Axiom needs a proof, since it does not hold for the geometry on a sphere.

(2) In order to bring before the perceptive intuition a geometry in which the triangle's angle-sum is less than the two right angles we need the help of an "imaginary sphere."

(3) In a space in which the triangle's angle-sum is different from two right angles, there is an absolute measure [a natural unit for length].

Just as Lambert's celebrated memoir, "Vorläufige Kenntnisse fuer die, so die Quadratur und Rectification des Circuls suchen," closed a question whose history comprises four thousand years, and which therefore pertains to the very oldest problems of mankind, so this essay of Lambert "Zur Theorie der Parallelinien" should have ended the equally fruitless and almost equally ancient striving after a proof of the Postulatum of Euclid, the ordinary assumption for the treatment of parallels, or anything equivalent to it. Will one man ever again wound to the death two such dragons feeding for centuries on human brains!

In a review of my translation of Vasiliev's Address on Lobachevski Mr. Charles S. Pierce says in *The Nation*, April 4th, 1895; "However, Gauss was not the first discoverer. Lambert in 1785, in a printed book, spoke plainly of a space where the angles of a triangle should sum up to less than 180 degrees, and mentions one of its most remarkable properties. Gauss most likely knew of this."

Should this be so, the last claim left to be made for Gauss in the determined and persistent endeavor of his German admirers to keep him prominently figuring in the history of the greatest achievement of modern culture, the non-Euclidean geometry, namely the claim that he was the first to recognize with complete clearness the uselessness of all attempts to prove the eleventh axiom, becomes meaningless.

---

## INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER. Ph. D., Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

(Continued from the June Number.)

In what precedes we have endeavored to convey some of the most important general concepts in regard to the theory of substitution groups, we proceed to some more special concepts leading up to the branch of this subject which we desire to pursue first, viz. the *construction* of the substitution groups containing a given number of letters.

We shall not aim at a rigorous development of any part of the subject but rather at conveying some of the most fruitful concepts and thus to aid (1) those who desire to form a good general idea of the subject without entering

into details and (2) those who desire to acquaint themselves with such facts and methods as may be helpful in a more thorough study. The methods used will in general, be such as lead to a rigorous development, but, for the sake of simplicity we shall not pursue them sufficiently far to call our treatment a rigorous development. "In presenting any mathematical theory it is in general, not so important that a rigorous presentation of the subject is given throughout as that the methods used are those which serve as a means for a rigorous presentation.\*

The theory of groups is divided into two parts, viz. *the theory of substitution groups* and *the theory of transformation groups*. To these we might perhaps add *the theory of abstract groups*, which was first suggested by Cayley. Its foundation was laid by Dyck in his noted articles in the *Mathematische Annalen* for 1882 and 1883. This theory has not yet been extensively developed. The theory of transformation groups was founded by Sophus Lie who began to publish his results in 1870. It bears the same relations to the solution of differential equations as the theory of substitution groups bears to the solution of the algebraic equations. It has not yet had time to take a definite station among the other branches of mathematics but the indications are that it will occupy a very high one.

The theory of substitution groups is considerably older than the two just named. It might be said to have been founded by Abel† but traces of it are found much earlier, as for instance in the writings of Lagrange.§ The most prominent among those who have contributed to its development are Cauchy, Galois and Jordan. Jordan's "Traite des Substitutions et des Equations" is the standard treatise. After this come Netto's "Substitutionentheorie und ihre Anwendungen auf die Algebra" and the chapters on this subject in Serret's "Cours d'Algebre Superieure," second volume. The only work on the subject in the English language is Professor Cole's translation of Netto's work. A great part of the knowledge of this, as well as of most other modern mathematical subjects, is to be found only in the journals.

Having found the place which the theory of substitution groups occupies in the general theory of groups we proceed to locate the theory of the construction of groups in the more general theory of substitution groups. Since groups are merely instruments of operation it follows that the study of the construction of groups must be the study of instruments and not the more useful study of their uses and the methods of using them. It is evident that the study of the use and the modes of using an instrument can, as a rule, be profitably pursued only after a thorough acquaintance had been formed with the instrument itself and it therefore seems proper for us first to pay attention to its study.

Another advantage in beginning in this way is the fact that a thorough

---

\* Weberhaupt dürfte es ja bei der Darlegung einer mathematischen Theorie weniger auf eine durchweg strenge Darstellung, als vielmehr darauf ankommen, dass die angegebenen methoden die zur strengen Darstellung erforderlichen mittel gewahren. Neumaan, S. VIII. Riemann's Theorie der Abel'schen Integrale.

†Crelle's Journal für Mathematik, vol. I. 1826.

§Dr. James Pierpont: Bulletin of the American Mathematical Society, May, 1895.

acquaintance with a complex instrument is apt to suggest uses and modes of using which we do not find fully described by others. In fact, substitution groups constitute such a complex instrument that it seems almost impossible to gain a thorough knowledge of it from descriptions alone, especially since descriptions, as a rule, are unattractive to those who do not already possess kindred concepts which require only slight modification.

We have already remarked that the process of finding one substitution which is equivalent to two successive substitutions is called multiplication and that the commutative law does not hold in this multiplication. On this account it is necessary to distinguish between multiplier and multiplicand by the order in which they are written. On this point there is no uniformity among the writers on this subject but we shall always suppose that the multiplier precedes the multiplicand. Thus we have

$$abc \cdot ab = bc.$$

For in the first substitution  $a$  is replaced by  $b$  and in the second this  $b$  is replaced by  $a$ , thus  $a$  remains unchanged. In the first substitution  $b$  is replaced by  $c$  and in the second substitution this  $c$  is unchanged, hence the two successive substitutions replace  $b$  by  $c$ . In the first substitution  $c$  is replaced by  $a$  and in the second this  $a$  is replaced by  $b$ , hence the two successive substitutions replace  $c$  by  $b$ . Take for example the expression

$$a + 3b + 2c.$$

After applying the first substitution this becomes

$$b + 3c + 2a.$$

If we now operate with the second substitution this becomes

$$a + 3c + 2b.$$

which is the same result as we should have obtained by operating upon the first expression by  $bc$ .

In constructing groups we shall have to use multiplication to a very large extent. The process, is however, very simple as may be inferred from the preceding example and others which have been given before. The great importance of this operation led us to introduce this additional example with explanations. Referring to the lists of groups of two, three, and four letters, we desire to call attention to an important property. Let us consider, for instance the two groups

$$\begin{array}{l} 1, ab \cdot cd, ac \cdot bd, ad \cdot bc \\ \text{and} \quad 1, ac, bd, ac \cdot bd \end{array}$$

In the first one we observe that each letter is replaced by every other letter; while in the second  $a$  is replaced by  $c$  but not by either  $b$  or  $d$ , similarly  $d$  is replaced by  $b$  but not by  $a$  or  $c$ . Groups, like the first, in which one letter is replaced by every other letter of the group are called *transitive* groups. Those, like the second, in which no letter is replaced by all of the others are called *intransitive* groups. In the preceding lists there are only two intransitive groups but when the number of letters exceeds five this class of groups is by far the larger of the two.

The methods used to construct the intransitive groups are much simpler than those used to construct the transitive, on this account we shall consider the construction of the intransitive groups first.

[To be continued.]